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# Analytic representation of finite quantum systems

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## Abstract

A transform between functions in  $\mathbb{R}$  and functions in  $\mathbb{Z}_d$  is used to define the analogue of number and coherent states in the context of finite  $d$ -dimensional quantum systems. The coherent states are used to define an analytic representation in terms of theta functions. All states are represented by entire functions with growth of order 2, which have exactly  $d$  zeros in each cell. The analytic function of a state is constructed from its zeros. Results about the completeness of finite sets of coherent states within a cell are derived.

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## 1. Introduction

Quantum systems with finite Hilbert space have been studied originally by Weyl and Schwinger [1] and later by many authors [2, 3]. A formalism analogous to the harmonic oscillator can be developed where the dual variables that we call ‘position’ and ‘momentum’ take values in  $\mathbb{Z}_d$  (the integers modulo  $d$ ). This area of research is interesting in its own right; and has many applications in areas such as quantum optics, quantum computing [4], two-dimensional electron systems in magnetic fields and the magnetic translation group [5], the quantum Hall effect [6], quantum maps [7], hydrodynamics [8] mathematical physics and signal processing, etc. For a review see [9].

In this paper we introduce a transform from functions in  $\mathbb{R}$  to functions in  $\mathbb{Z}_d$ . This is related to Zak transform from functions in  $\mathbb{R}$  to functions on a circle [10–12]; but of course here we have functions in ‘discretized circle’. This transform enables us to transfer some of the harmonic oscillator formalism into the context of finite systems. For example, we define the analogues of number states and coherent states. Coherent states in the context of finite systems have been previously considered in [13, 14].

Coherent states can be used to define analytic representations. For example, ordinary coherent states of the harmonic oscillator can be used to define the Bargmann analytic representation in the complex plane;  $SU(1, 1)$  coherent states can be used to define analytic representations in the unit disc (Lobachevsky geometry);  $SU(2)$  coherent states can be used to define analytic representations in the extended complex plane (spherical geometry). We use

the coherent states in the context of finite systems to define analytic representations. Similar analytic representations have been used in the context of quantum maps in [13]. We show that the corresponding analytic functions obey doubly periodic boundary conditions; and therefore it is sufficient to define them on a square cell  $S$ . Each of these analytic functions has growth of order 2 and has exactly  $d$  zeros in  $S$ .

We use the analytic formalism to study the completeness of finite sets of coherent states in the cell  $S$ . This discussion is the analogue in the present context, of the ‘theory of von Neumann lattice’ for the harmonic oscillator [15–18], which is based on the theory of the density of zeros of analytic functions [19].

In section 2, we review briefly the basic theory of finite systems and define some quantities for later use. In section 3, we introduce the transform between functions in  $\mathbb{R}$  and functions in  $\mathbb{Z}_d$ . Using this transform we define in section 4 number states and coherent states in our context of finite systems, and study their properties. In section 5, we use the coherent states to define an analytic representation in terms of theta functions. We show that the order of the growth of these entire functions is 2. We also study displacements and the Heisenberg–Weyl group in this language. In section 6, we study the zeros of the corresponding analytic functions and use them to study the completeness of finite sets of coherent states within a cell. In section 7, we construct the analytic representation of a state from its zeros. We conclude in section 8 with the discussion of our results.

## 2. Finite quantum systems

### 2.1. Position and momentum states and Fourier transform

We consider a quantum system with a  $d$ -dimensional Hilbert space  $\mathcal{H}$ . We use the notation  $|s\rangle\rangle$  for the states in  $\mathcal{H}$ ; and we use the notation  $|s\rangle$  for the states in the infinite-dimensional Hilbert space  $H$  associated with the harmonic oscillator. Let  $|X; m\rangle\rangle$  be an orthonormal basis in  $\mathcal{H}$ , where  $m$  belongs to  $\mathbb{Z}_d$ . We refer to them as ‘position states’. The  $X$  in the notation is not a variable but it simply indicates position states.

The finite Fourier transform is defined as

$$F = d^{-1/2} \sum_{m,n} \omega(mn) |X; m\rangle\rangle \langle\langle X; n|; \quad \omega(\alpha) = \exp\left(i \frac{2\pi\alpha}{d}\right) \quad (1)$$

$$FF^\dagger = F^\dagger F = \mathbf{1}; \quad F^4 = \mathbf{1}. \quad (2)$$

Using the Fourier transform we define another orthonormal basis, the ‘momentum states’, as

$$|P; m\rangle\rangle = F |X; m\rangle\rangle = d^{-1/2} \sum_{n=0}^{d-1} \omega(mn) |X; n\rangle\rangle. \quad (3)$$

We also define the ‘position and momentum operators’  $x$  and  $p$  as

$$x = \sum_{n=0}^{d-1} n |X; n\rangle\rangle \langle\langle X; n|; \quad p = \sum_{n=0}^{d-1} n |P; n\rangle\rangle \langle\langle P; n|. \quad (4)$$

It is easily seen that

$$Fx F^\dagger = p; \quad Fp F^\dagger = -x. \quad (5)$$

## 2.2. Displacements and the Heisenberg–Weyl group

The displacement operators are defined as

$$Z = \exp\left(i\frac{2\pi}{d}x\right); \quad X = \exp\left(-i\frac{2\pi}{d}p\right) \quad (6)$$

$$X^d = Z^d = \mathbf{1}; \quad X^\beta Z^\alpha = Z^\alpha X^\beta \omega(-\alpha\beta) \quad (7)$$

where  $\alpha, \beta$  are integers in  $\mathbb{Z}_d$ . They perform displacements along the  $P$  and  $X$  axes in the  $X$ – $P$  phase space. Indeed we can show that

$$Z^\alpha |P; m\rangle = |P; m + \alpha\rangle; \quad Z^\alpha |X; m\rangle = \omega(\alpha m) |X; m\rangle \quad (8)$$

$$X^\beta |P; m\rangle = \omega(-m\beta) |P; m\rangle; \quad X^\beta |X; m\rangle = |X; m + \beta\rangle. \quad (9)$$

The  $X$ – $P$  phase space is the toroidal lattice  $\mathbb{Z}_d \times \mathbb{Z}_d$ .

The general displacement operators are defined as

$$D(\alpha, \beta) = Z^\alpha X^\beta \omega(-2^{-1}\alpha\beta); \quad [D(\alpha, \beta)]^\dagger = D(-\alpha, -\beta). \quad (10)$$

It is easy to see

$$D(\alpha, \beta) |X; m\rangle = \omega(2^{-1}\alpha\beta + \alpha m) |X; m + \beta\rangle \quad (11)$$

$$D(\alpha, \beta) |P; m\rangle = \omega(-2^{-1}\alpha\beta - \beta m) |P; m + \alpha\rangle.$$

We next consider an arbitrary (normalized) state  $|s\rangle$

$$|s\rangle = \sum_{m=0}^{d-1} s_m |X; m\rangle; \quad \sum_{m=0}^{d-1} |s_m|^2 = 1 \quad (12)$$

and act with the displacement operators to get the states

$$|s; \alpha, \beta\rangle \equiv D(\alpha, \beta) |s\rangle = \sum_{m=0}^{d-1} s_m \omega(2^{-1}\alpha\beta + \alpha m) |X; m + \beta\rangle. \quad (13)$$

Clearly  $|s; 0, 0\rangle = |s\rangle$ . Using equations (8) and (9) we easily show that

$$d^{-1} \sum_{\alpha, \beta=0}^{d-1} |s; \alpha, \beta\rangle \langle s; \alpha, \beta| = \mathbf{1}_d. \quad (14)$$

This shows that the states  $|s; \alpha, \beta\rangle$  (for a fixed ‘fiducial’ state  $|s\rangle$ ) and all  $\alpha, \beta$  in  $\mathbb{Z}_d$  form an overcomplete basis of  $d^2$  vectors in the  $d$ -dimensional Hilbert space  $\mathcal{H}$ . Equation (14) is the resolution of the identity.

## 2.3. General transformations

In this section we expand an arbitrary operator  $\Omega$ , in terms of displacement operators. In order to do this we first define its Weyl function as

$$\tilde{W}_\Omega(\alpha, \beta) = \text{Tr}[\Omega D(\alpha, \beta)]. \quad (15)$$

The properties of the Weyl function and its relation to the Wigner function is discussed in [9]. We can prove that

$$\Omega = d^{-1} \sum_{\alpha, \beta=0}^{d-1} \tilde{W}_\Omega(-\alpha, -\beta) D(\alpha, \beta). \quad (16)$$

### 3. A transform between functions in $\mathbb{R}$ and functions in $\mathbb{Z}_d$

In this section we introduce a map between states in the infinite-dimensional harmonic oscillator Hilbert space  $H$  and the  $d$ -dimensional Hilbert space  $\mathcal{H}$ . This map is a special case of the Zak transform. We consider a state  $|\psi\rangle$  in  $H$  with (normalized) wavefunction in the  $x$ -representation  $\psi(x) = \langle x|\psi\rangle$ . The corresponding state  $|\psi\rangle\rangle$  in  $\mathcal{H}$  is defined through the map

$$\psi_m = \langle\langle X; m|\psi\rangle\rangle = \mathcal{N}^{-1/2} \sum_{w=-\infty}^{\infty} \psi \left[ x = \left( \frac{2\pi}{d} \right)^{1/2} \lambda(m + dw) \right]; \quad \psi_{m+d} = \psi_m \quad (17)$$

where  $m \in \mathbb{Z}_d$ .  $\mathcal{N}$  is a normalization factor so that  $\sum_{m=0}^{d-1} |\psi_m|^2 = 1$  which is given in appendix A. The Fourier transform (on the real line) of  $\psi(x)$  is defined as

$$\tilde{\psi}(p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \psi(x) \exp(-ipx) dx. \quad (18)$$

Using the map of equation (17) we define

$$\tilde{\psi}_m = \mathcal{N}'^{-1/2} \sum_{w=-\infty}^{\infty} \tilde{\psi} \left[ p = \left( \frac{2\pi}{d} \right)^{1/2} \frac{1}{\lambda}(m + dw) \right]. \quad (19)$$

The tilde in  $\tilde{\psi}$  indicates that the Fourier transform of  $\psi(x)$  has been transformed according to equation (18). The normalization factor  $\mathcal{N}'$  is given in appendix A where it is shown that  $\mathcal{N}' = \lambda^2 \mathcal{N}$ .

We next prove that

$$\tilde{\psi}_m = d^{-1/2} \sum_{n=0}^{d-1} \omega(-mn) \psi_n = \langle\langle P; m|\psi\rangle\rangle. \quad (20)$$

This shows that  $\tilde{\psi}_m$  is the finite Fourier transform of  $\psi_n$ , and therefore the tilde also indicates the above finite Fourier transform. So the tilde in the notation is used for two different Fourier transforms, but they are consistent to each other.

In order to prove equation (20) we insert equation (17) into equation (20) and use the Poisson formula

$$\sum_{w=-\infty}^{\infty} \exp(i2\pi wx) = \sum_{k=-\infty}^{\infty} \delta(x - k), \quad (21)$$

where the right-hand side is the 'comb delta function'; and also the relation

$$\frac{1}{d} \sum_{m=0}^{d-1} \omega[m(k - \ell)] = \delta(k, \ell); \quad k, \ell \in \mathbb{Z}_d \quad (22)$$

where  $\delta(k, k')$  is a Kronecker delta. These two relations are useful in many proofs in this paper.

The above map is not one-to-one (the Hilbert space  $H$  is infinite dimensional while the Hilbert space  $\mathcal{H}$  is  $d$ -dimensional). Therefore, equation (17) cannot be inverted. In appendix B, we use the full Zak transform and introduce a family of  $d$ -dimensional Hilbert spaces  $\mathcal{H}(\sigma_1, \sigma_2)$  with twisted boundary conditions. We show that the Hilbert space  $H$  is isomorphic to the direct integral of all the  $\mathcal{H}(\sigma_1, \sigma_2)$  (with  $0 \leq \sigma_1 < 1, 0 \leq \sigma_2 < 1$ ) and then an inverse to the relation (17) can be found. However, the formalism of this paper is valid only for  $\mathcal{H} \equiv \mathcal{H}(0, 0)$  (which has periodic boundary conditions).

**Table 1.** Eigenvectors of the Fourier operator  $F$  with  $d = 6$ .

$ 0\rangle\rangle$	$ 1\rangle\rangle$	$ 2\rangle\rangle$	$ 3\rangle\rangle$	$ 4\rangle\rangle$	$ 6\rangle\rangle$
0.759 71	0	-0.525 46	0	0.370 40	-0.314 49
0.450 04	0.653 28	0.340 71	-0.270 59	-0.378 23	0.285 78
0.093 73	0.270 60	0.481 31	0.653 28	0.374 71	-0.158 03
0.013 65	0	0.168 51	0	0.543 93	0.829 34
0.093 73	-0.270 60	0.481 31	-0.653 28	0.374 71	-0.158 03
0.450 04	-0.653 28	0.340 71	0.270 59	-0.378 23	0.285 78

#### 4. Quantum states

##### 4.1. Number eigenstates

In the harmonic oscillator, number states are eigenstates of the Fourier operator  $\exp(ia^\dagger a)$  where  $a, a^\dagger$  are the usual annihilation and creation operators. In this section we apply the transformation of equation (17) with  $\lambda = 1$  and we show that the resulting states are eigenstates of the Fourier operator of equation (1).

We consider the harmonic oscillator number eigenstates  $|N\rangle$  whose wavefunction is

$$\chi(x, N) = \langle x|N\rangle = \left(\frac{1}{\pi^{1/2}2^N N!}\right)^{1/2} \exp\left(-\frac{1}{2}x^2\right) H_N(x) \tag{23}$$

where  $H_N(x)$  are Hermite polynomials.

It is known that

$$\tilde{\chi}(x, N) = i^N \chi(x, N). \tag{24}$$

Using the transforms of equations (17) and (19) with  $\lambda = 1$ , we find

$$\chi_m(N) = \langle\langle X; m|N\rangle\rangle = \mathcal{N}_n(N)^{-1/2} \sum_{w=-\infty}^{\infty} \chi\left[x = (m + dw) \left(\frac{2\pi}{d}\right)^{1/2}, N\right], \tag{25}$$

$$\tilde{\chi}_m(N) = \langle\langle P; m|N\rangle\rangle = \mathcal{N}_n(N)^{-1/2} \sum_{w=-\infty}^{\infty} \tilde{\chi}\left[x = (m + dw) \left(\frac{2\pi}{d}\right)^{1/2}, N\right], \tag{26}$$

where  $\mathcal{N}_n(N)$  is the normalization factor for number eigenstates, given by equation (A.1) with  $\psi$  replaced by  $\chi$ . Equation (24) implies that

$$\tilde{\chi}_m(N) = i^N \chi_m(N). \tag{27}$$

Using this in conjunction with equation (20) we prove that

$$d^{-1/2} \sum_{n=0}^{d-1} \omega(-mn) \chi_n(N) = i^N \chi_m(N); \quad \rightarrow \quad F|N\rangle\rangle = i^N |N\rangle\rangle. \tag{28}$$

Therefore the vectors  $\chi_m(N)$  are eigenvectors of the Fourier matrix. They have been studied in the context of signal processing in [20]. Of course, the Fourier matrix is finite and only  $d$  of these eigenvectors are linearly independent. Therefore, the set of all states  $|N\rangle\rangle$  is highly overcomplete. In general, the number states  $|N\rangle\rangle$  are not orthogonal to each other.

The Fourier matrix has four eigenvalues  $i^k$  ( $0 \leq k \leq 3$ ); and all the states  $|N = 4M + k\rangle\rangle$  correspond to the same eigenvalue  $i^k$ . As an example, we consider the case  $d = 6$  and using equation (25) we calculate the six eigenvectors. Results are presented in table 1 (we note that  $|5\rangle\rangle = -|1\rangle\rangle$ ).

#### 4.2. Coherent states

We consider the harmonic oscillator coherent states  $|A\rangle$  whose wavefunction is

$$\psi(x, A) = \langle x|A\rangle = \pi^{-1/4} \exp\left(-\frac{1}{2}x^2 + Ax - \frac{1}{2}A_R A\right), \quad (29)$$

where  $A = A_R + iA_I$ . Using the transformation of equation (17) we introduce coherent states  $|A\rangle\rangle$  in the finite Hilbert space as

$$\begin{aligned} \psi_m(A) &= \langle\langle X; m|A\rangle\rangle = \mathcal{N}_C(A)^{-1/2} \pi^{-1/4} \exp\left[-\frac{\pi\lambda^2 m^2}{d} + Am\lambda \left(\frac{2\pi}{d}\right)^{1/2} - \frac{1}{2}A_R A\right] \\ &\quad \times \Theta_3\left[i\pi m\lambda^2 - iA\lambda \left(\frac{d\pi}{2}\right)^{1/2}; id\lambda^2\right] \\ &= \mathcal{N}_C(A)^{-1/2} \pi^{-1/4} d^{-1/2} \lambda^{-1} \exp\left(\frac{i}{2}A_I A\right) \cdot \Theta_3\left[\frac{\pi m}{d} - \frac{A}{\lambda} \left(\frac{\pi}{2d}\right)^{1/2}; \frac{i}{d\lambda^2}\right], \end{aligned} \quad (30)$$

where  $\Theta_3$  are theta functions [21], defined as

$$\Theta_3(u; \tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi\tau n^2 + i2nu), \quad (31)$$

and the relation

$$\Theta_3(u; \tau) = (-i\tau)^{-1/2} \exp\left(\frac{u^2}{\pi i\tau}\right) \cdot \Theta_3\left(\frac{u}{\tau}; -\frac{1}{\tau}\right) \quad (32)$$

has been used in equation (30). The normalization factor, in the case of even dimension  $d = 2k$ , is

$$\mathcal{N}_C(A) = \left(\frac{d}{2\pi}\right)^{1/2} \lambda^{-1} \cdot \Theta_3\left[A_I\lambda \left(\frac{\pi d}{2}\right)^{1/2}; \frac{id\lambda^2}{2}\right] \cdot \Theta_3\left[\frac{A_R}{\lambda} \left(\frac{\pi d}{2}\right)^{1/2}; \frac{id}{2\lambda^2}\right] \quad (33)$$

and in the case of odd dimension  $d = 2k + 1$  it is

$$\mathcal{N}_C(A) = \left(\frac{d}{2\pi}\right)^{1/2} \lambda^{-1} \cdot \Theta_3\left[A_I\lambda \left(\frac{\pi d}{2}\right)^{1/2}; \frac{id\lambda^2}{2}\right] \cdot \Theta_3\left[\frac{A_R}{\lambda} (2\pi d)^{1/2}; \frac{2id}{\lambda^2}\right]. \quad (34)$$

Equations (33) and (34) can be written in the single form

$$\begin{aligned} \mathcal{N}_C(A) &= \left(\frac{d}{2\pi}\right)^{1/2} \lambda^{-1} \cdot \Theta_3\left[A_I\lambda \left(\frac{\pi d}{2}\right)^{1/2}; \frac{id\lambda^2}{2}\right] \\ &\quad \times \Theta_3\left[\frac{A_R}{\lambda} \left(\frac{4\pi d}{5 + 3(-1)^d}\right)^{1/2}; \frac{4id}{[5 + 3(-1)^d]\lambda^2}\right]. \end{aligned} \quad (35)$$

The  $\psi_m(A)$  obeys the relations

$$\begin{aligned} \psi_m[A + (2\pi d)^{1/2}\lambda] &= \psi_m(A) \exp\left[iA_I\lambda \left(\frac{\pi d}{2}\right)^{1/2}\right]; \\ \psi_m\left[A + i\frac{(2\pi d)^{1/2}}{\lambda}\right] &= \psi_m(A) \exp\left[-i\frac{A_R}{\lambda} \left(\frac{\pi d}{2}\right)^{1/2}\right]. \end{aligned} \quad (36)$$

The zeros of the theta function  $\Theta_3(u; \tau)$  are given by

$$u = (2k + 1)\frac{\pi}{2} + (2l + 1)\frac{\pi\tau}{2}, \quad (37)$$

where  $k, l$  are integers. Therefore,

$$\psi_m(A_{kl}) = \langle\langle X; m | A_{kl}(m) \rangle\rangle = 0; \quad A_{kl}(m) = \left(\frac{2\pi}{d}\right)^{1/2} \left[ \left(kd + \frac{d}{2} + m\right) \lambda + \frac{(2l+1)i}{2\lambda} \right]. \tag{38}$$

It is seen that the states  $|A_{kl}(m)\rangle$  are orthogonal to the position states  $|X; m\rangle$ .

The ‘vacuum state’  $|0\rangle$  is defined as

$$\langle\langle X; m | 0 \rangle\rangle = \mathcal{N}_C(0)^{-1/2} \pi^{-1/4} d^{-1/2} \cdot \Theta_3\left(\frac{\pi m}{d}; \frac{i}{d\lambda^2}\right), \tag{39}$$

where

$$\mathcal{N}_C(0) = \left(\frac{2d}{\pi}\right)^{1/2} \lambda^{-1} \cdot \Theta_3\left(0; \frac{id\lambda^2}{2}\right) \Theta_3\left(0; \frac{4id}{[5+3(-1)^d]\lambda^2}\right). \tag{40}$$

The coherent states  $|A\rangle$  satisfy the following resolution of the identity:

$$\lambda(2\pi d)^{-1/2} \int_S d^2A \mathcal{N}_C(A) |A\rangle \langle\langle A | = \mathbf{1}_d; \tag{41}$$

$$S = \left[ a, a + (2\pi d)^{1/2} \lambda \right]_R \times \left[ b, b + \frac{(2\pi d)^{1/2}}{\lambda} \right]_I.$$

We integrate here over the cell  $S$ . The periodicity of equation (36) implies that the cell can be shifted everywhere in the complex plane and this is indicated with the arbitrary real numbers  $a, b$ . The proof of equation (41) is based on the resolution of the identity for ordinary (harmonic oscillator) coherent states, in conjunction with the map of equation (17).

The set of all coherent states in the cell  $S$  is highly overcomplete. Indeed using equation (14) we easily show another resolution of identity which involves only  $d^2$  coherent states in the cell  $S$ :

$$d^{-1} \sum_{\alpha, \beta=0}^{d-1} \left| A + \left(\frac{2\pi}{d}\right)^{1/2} \left(\beta\lambda + \frac{\alpha}{\lambda}i\right) \right\rangle \langle\langle A + \left(\frac{2\pi}{d}\right)^{1/2} \left(\beta\lambda + \frac{\alpha}{\lambda}i\right) | = \mathbf{1}_d. \tag{42}$$

The overlap of two coherent states is

$$\begin{aligned} \langle\langle A_1 | A_2 \rangle\rangle &= \left(\frac{d}{2\pi}\right)^{1/2} \lambda^{-1} \mathcal{N}_C(A_1)^{-1/2} \mathcal{N}_C(A_2)^{-1/2} \exp\left(-\frac{1}{4}|A_1|^2 - \frac{1}{4}|A_2|^2 + \frac{1}{2}A_1^* A_2\right) \\ &\times \Theta_3\left[ i(A_2 - A_1^*)\lambda \left(\frac{\pi d}{8}\right)^{1/2}; \frac{id\lambda^2}{2} \right] \\ &\times \Theta_3\left[ \frac{A_2 + A_1^*}{\lambda} \left(\frac{\pi d}{5+3(-1)^d}\right)^{1/2}; \frac{4id}{[5+3(-1)^d]\lambda^2} \right]. \end{aligned} \tag{43}$$

The first theta function in the above relation is zero when

$$A_2 - A_1^* = \left(l + \frac{1}{2}\right) (2\pi d)^{1/2} \lambda + i(2k+1) \left(\frac{2\pi}{d}\right)^{1/2} \lambda^{-1} \tag{44}$$

and the second is zero when

$$A_2 + A_1^* = \left(\frac{[5+3(-1)^d]\pi}{4d}\right)^{1/2} \lambda(2k+1) + \left(\frac{4\pi d}{5+3(-1)^d}\right)^{1/2} \lambda^{-1} (2l+1)i. \tag{45}$$

The corresponding coherent states in these two cases are orthogonal to each other.



There is a relation between the coherent states in a finite Hilbert space studied in this section and the number states studied earlier:

$$|A\rangle\rangle = \exp\left(-\frac{|A|^2}{2}\right) \sum_{N=0}^{\infty} \frac{A^N}{\sqrt{N!}} \left[\frac{\mathcal{N}_n(N)}{\mathcal{N}_C(A)}\right]^{1/2} |N\rangle\rangle. \quad (46)$$

This is analogous to the relation between coherent states and number states in the infinite-dimensional Hilbert space for the harmonic oscillator. We have explained earlier that only  $d$  of the number states appearing in the right-hand side of equation (46) are independent.

Introducing the displacement operator defined in equation (10), we can prove that

$$D(\alpha, \beta)|A\rangle\rangle = \left|A + \left(\frac{2\pi}{d}\right)^{1/2} (\beta\lambda + \alpha\lambda^{-1}i)\right\rangle\rangle \exp\left[-iA_I\lambda\left(\frac{\pi}{2d}\right)^{1/2}\beta + iA_R\lambda^{-1}\left(\frac{\pi}{2d}\right)^{1/2}\alpha\right], \quad (47)$$

where both  $\alpha$  and  $\beta$  are integers. We might be tempted to use the above equation as a definition for displacement operators with real values of  $\alpha, \beta$ . It can be shown that in this case  $D$  depends on  $A$ ; and only for integer  $\alpha, \beta$  the  $D$  is independent of  $A$ .

## 5. Analytic representation

### 5.1. Quantum states

Let  $|f\rangle\rangle$  be an arbitrary (normalized) state

$$|f\rangle\rangle = \sum_{m=0}^{d-1} f_m |X; m\rangle\rangle; \quad \sum_{m=0}^{d-1} |f_m|^2 = 1. \quad (48)$$

We shall use the notation

$$|f^*\rangle\rangle = \sum_{m=0}^{d-1} f_m^* |X; m\rangle\rangle; \quad \langle\langle f| = \sum_{m=0}^{d-1} f_m^* \langle\langle X; m|; \quad \langle\langle f^*| = \sum_{m=0}^{d-1} f_m \langle\langle X; m|. \quad (49)$$

We define the analytic representation of  $|f\rangle\rangle$  as

$$\begin{aligned} f(z) &\equiv \mathcal{N}_C(z)^{1/2} d^{1/2} \lambda \exp\left(-\frac{i}{2} z_I z\right) \langle\langle z^* | f \rangle\rangle \\ &= \pi^{-1/4} \sum_{m=0}^{d-1} \Theta_3 \left[ \frac{\pi m}{d} - \frac{z}{\lambda} \left(\frac{\pi}{2d}\right)^{1/2}; \frac{i}{d\lambda^2} \right] f_m \end{aligned} \quad (50)$$

where  $|z\rangle\rangle$  is a coherent state. It is easy to see

$$f[z + (2\pi d)^{1/2} \lambda] = f(z); \quad f[z + i(2\pi d)^{1/2} \lambda^{-1}] = f(z) \exp\left[\frac{\pi d}{\lambda^2} - i(2\pi d)^{1/2} z \lambda^{-1}\right]. \quad (51)$$

The  $f(z)$  is an entire function. If  $M(R)$  is the maximum modulus of  $f(z)$  for  $|z| = R$ , then

$$\rho = \limsup_{R \rightarrow \infty} \frac{\ln \ln M(R)}{\ln R} \quad (52)$$

is the order of the growth of  $f(z)$  [19]. It is easily seen that in our case the order of the growth is  $\rho = 2$ .

Due to the periodicity our discussion below is limited to a single cell  $S$  (defined in equation (41)). The scalar product is given by

$$\langle\langle f^* | g \rangle\rangle = (2\pi)^{-1/2} d^{-3/2} \lambda^{-1} \int_S d^2 z \exp(-z_I^2) f(z) g(z^*). \quad (53)$$

As special cases, we derive the analytic representations of the position states:

$$|X; m\rangle \rightarrow \pi^{-1/4} \cdot \Theta_3 \left[ \frac{\pi m}{d} - z\lambda^{-1} \left( \frac{\pi}{2d} \right)^{1/2}; \frac{i}{d\lambda^2} \right], \tag{54}$$

momentum states:

$$|P; m\rangle \rightarrow \lambda\pi^{-1/4} \exp \left( -\frac{1}{2}z^2 \right) \cdot \Theta_3 \left[ \frac{\pi m}{d} - \lambda zi \left( \frac{\pi}{2d} \right)^{1/2}; \frac{i\lambda^2}{d} \right] \tag{55}$$

and the coherent states:

$$\begin{aligned} |A\rangle \rightarrow f(z, A) &= d \left[ \frac{2}{\pi\mathcal{N}_C(A)} \right]^{1/2} \exp \left( -\frac{1}{4}z^2 + \frac{1}{2}Az - \frac{1}{4}|A|^2 \right) \\ &\times \Theta_3 \left[ i(A-z)\lambda \left( \frac{\pi d}{8} \right)^{1/2}; \frac{id\lambda^2}{2} \right] \\ &\times \Theta_3 \left[ (A+z)\lambda^{-1} \left( \frac{\pi d}{5+3(-1)^d} \right)^{1/2}; \frac{4id}{[5+3(-1)^d]\lambda^2} \right]. \end{aligned} \tag{56}$$

### 5.2. Displacements and the Heisenberg–Weyl group

In this section we express the displacement operators  $X$  and  $Z$  in the context of analytic representations. Equations (9) and (8) are written as

$$\begin{aligned} Xf(z) &= f \left[ z - \left( \frac{2\pi}{d} \right)^{1/2} \lambda \right]; \\ Zf(z) &= f \left[ z + i \left( \frac{2\pi}{d} \right)^{1/2} \lambda^{-1} \right] \exp \left[ iz\lambda^{-1} \left( \frac{2\pi}{d} \right)^{1/2} - \frac{\pi}{d\lambda^2} \right]. \end{aligned} \tag{57}$$

Therefore,  $X$  and  $Z$  are given by

$$\begin{aligned} X &= \exp \left[ - \left( \frac{2\pi}{d} \right)^{1/2} \lambda \partial_z \right] \\ Z &= \exp \left[ iz\lambda^{-1} \left( \frac{2\pi}{d} \right)^{1/2} - \frac{\pi}{d\lambda^2} \right] \exp \left[ i \left( \frac{2\pi}{d} \right)^{1/2} \lambda^{-1} \partial_z \right] \end{aligned} \tag{58}$$

and the general displacement operator is

$$D(\alpha, \beta) = \omega(-2^{-1/2}\alpha\beta) \exp \left[ i\alpha z\lambda^{-1} \left( \frac{2\pi}{d} \right)^{1/2} - \frac{\alpha^2\pi}{d\lambda^2} \right] \exp \left[ (i\alpha\lambda^{-1} - \beta\lambda) \left( \frac{2\pi}{d} \right)^{1/2} \partial_z \right] \tag{59}$$

where  $\alpha, \beta$  are integers in  $\mathbb{Z}_d$ . Acting with this operator on the state  $|f\rangle$  of equation (48) represented by the analytic function  $f(z)$  of equation (50), we get

$$\begin{aligned} D(\alpha, \beta)f(z) &= \pi^{-1/4} \exp \left[ i\alpha z\lambda^{-1} \left( \frac{2\pi}{d} \right)^{1/2} - \frac{\alpha^2\pi}{d\lambda^2} - \frac{2^{1/2}\alpha\beta\pi i}{d} \right] \\ &\times \sum_{m=0}^{d-1} f_m \Theta_3 \left[ \frac{\pi m}{d} - \frac{z}{\lambda} \left( \frac{\pi}{2d} \right)^{1/2} - \frac{\alpha\pi i}{d\lambda} + \frac{\beta\pi}{d}; \frac{i}{d\lambda^2} \right]. \end{aligned} \tag{60}$$

### 5.3. General transformations

We have seen in equation (16) that an arbitrary operator  $\Omega$  can be expanded in terms of displacement operators and using this we can express  $\Omega$  as

$$\Omega = d^{-1} \sum_{\alpha, \beta=0}^{d-1} \omega(-2^{-1/2}\alpha\beta) \tilde{W}_{\Omega}(-\alpha, -\beta) \times \exp \left[ i\alpha z \lambda^{-1} \left( \frac{2\pi}{d} \right)^{1/2} - \frac{\alpha^2 \pi}{d\lambda^2} \right] \exp \left[ (i\alpha \lambda^{-1} - \beta \lambda) \left( \frac{2\pi}{d} \right)^{1/2} \partial_z \right]. \quad (61)$$

Alternatively, the operator  $\Omega = \sum_{m,n} \Omega_{mn} |X; m\rangle \langle X; n|$  can be represented with the kernel

$$\begin{aligned} \Omega(z, \zeta^*) &\equiv \mathcal{N}_C(\zeta)^{1/2} \mathcal{N}_C(z)^{1/2} \lambda^2 \exp \left( -\frac{i}{2} z_I z + \frac{i}{2} \zeta_I \zeta^* \right) \langle \zeta^* | \Omega | \zeta^* \rangle \\ &= \pi^{-1/2} d^{-1} \sum_{m,n=0}^{d-1} \Omega_{mn} \Theta_3 \left[ \frac{\pi m}{d} - \frac{z}{\lambda} \left( \frac{\pi}{2d} \right)^{1/2}; \frac{i}{d\lambda^2} \right] \\ &\quad \times \Theta_3 \left[ \frac{\pi n}{d} - \frac{\zeta^*}{\lambda} \left( \frac{\pi}{2d} \right)^{1/2}; \frac{i}{d\lambda^2} \right] \end{aligned} \quad (62)$$

and we easily prove that

$$\Omega |f\rangle \rightarrow (2\pi d)^{-1/2} \lambda^{-1} \int_S d^2 \zeta \exp(-\zeta_I \zeta^*) \Omega(z, \zeta^*) f(\zeta). \quad (64)$$

It is easily seen that

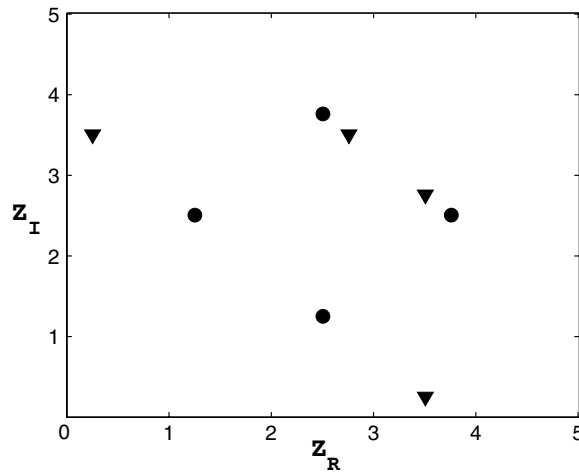
$$\begin{aligned} \Omega[z + (2\pi d)^{-1/2} \lambda \alpha, \zeta^* + (2\pi d)^{-1/2} \lambda \beta] &= \Omega[z, \zeta^*] \\ \Omega[z + i(2\pi d)^{-1/2} \lambda^{-1} \alpha, \zeta^*] &= \Omega[z, \zeta^*] \exp \left[ \frac{\pi d}{\lambda^2} \alpha^2 - i(2\pi d)^{1/2} z \alpha \lambda^{-1} \right] \\ \Omega[z, \zeta^* + i(2\pi d)^{-1/2} \lambda^{-1} \beta] &= \Omega[z, \zeta^*] \exp \left[ \frac{\pi d}{\lambda^2} \beta^2 - i(2\pi d)^{1/2} \zeta^* \beta \lambda^{-1} \right] \end{aligned} \quad (65)$$

where  $\alpha$  and  $\beta$  are integers. As an example, we consider the displacement operator  $X$  and find that

$$\begin{aligned} X(z, \zeta^*) &= \left( \frac{d}{2\pi} \right)^{1/2} \exp \left[ -\frac{1}{4} z^2 - \frac{1}{4} \zeta^{*2} - \frac{1}{2} z \zeta^* + \left( \frac{\pi}{2d} \right)^{1/2} (z - \zeta^*) - \frac{\pi}{2d} \right] \\ &\quad \times \Theta_3 \left[ \frac{\pi i}{2} - (\zeta^* - z) \lambda i \left( \frac{\pi d}{8} \right)^{1/2}; \frac{id\lambda^2}{2} \right] \\ &\quad \times \Theta_3 \left\{ [5 + 3(-1)^d]^{-1/2} [\sqrt{2}\pi - (\pi d)^{1/2} (z + \zeta^*) \lambda^{-1}]; \frac{4id}{[5 + 3(-1)^d] \lambda^2} \right\}. \end{aligned} \quad (66)$$

## 6. Zeros of the analytic representation and their physical meaning

If  $z_0$  is a zero of the analytic representation  $f(z)$ , then equation (50) shows that the coherent state  $|z_0\rangle$  is orthogonal to the state  $|f\rangle$ .



**Figure 1.** The zeros within a cell of the coherent states  $|0\rangle\rangle$  (circles) and  $|1 + i\rangle\rangle$  (triangles) for the case  $d = 4$ .

Using the periodicity of equation (51) we easily prove that

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f'(z)}{f(z)} dz = d, \tag{67}$$

where  $\Gamma$  is the boundary of the cell  $S$ . The above integral is in general equal to the number of zeros minus the number of poles of the function  $f(z)$  inside  $\Gamma$ . Since our functions have no poles, we conclude that the analytic representation of any state has exactly  $d$  zeros in the square  $S$  (zeros will be counted with their multiplicities). The area of  $S$  is  $2\pi d$ , and therefore there is an average of one zero per  $2\pi$  area of the complex plane, in this analytic representation. As an example we show in figure 1 the zeros of the coherent states  $|0\rangle\rangle$  and  $|1 + i\rangle\rangle$  for the case  $d = 4$ .

A direct consequence of this result is the fact that any set of  $d + 1$  coherent states in the cell  $S$  is at least complete. Indeed if it is not complete, then there exists some state which is orthogonal to all these coherent states. But such a state would have  $d + 1$  zeros, which is not possible. A set of  $d + 1$  states in a  $d$ -dimensional space which is at least complete is in fact overcomplete; in the sense that there exist a state which we can take out and be left with a complete set of  $d$  states. We note that if we take out an arbitrary state we might be left with an undercomplete set of  $d$  states.

A set of  $d - 1$  coherent states is clearly undercomplete, because our Hilbert space is  $d$  dimensional.

A set of  $d$  distinct coherent states  $\{|z_i\rangle\rangle; i = 1, \dots, d\}$  will be complete or undercomplete depending on whether it violates or satisfies the constraint

$$\sum_{i=1}^d z_i = \left(\frac{\pi}{2}\right)^{1/2} d^{3/2}(\lambda + i\lambda^{-1}) + (2\pi d)^{1/2}(M\lambda + iN\lambda^{-1}), \tag{68}$$

where  $M, N$  are integers. In order to prove this we use the periodicity of equation (51) to prove that

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{zf'(z)}{f(z)} dz = \left(\frac{\pi}{2}\right)^{1/2} d^{3/2}(\lambda + i\lambda^{-1}) + (2\pi d)^{1/2}(M\lambda + iN\lambda^{-1}). \tag{69}$$

The above integral is in general equal to the sum of zeros minus the sum of poles (with the multiplicities taken into account) of the function  $f(z)$  inside  $\Gamma$ . Since our functions have no poles, we conclude that the sum of zeros is equal to the right-hand side of equation (69). Equations (67) and (69) have also been given in [13].

If the  $d$  coherent states considered violate equation (68), then clearly they form a complete set because there exists no state which is orthogonal to all of them. If however they do satisfy the constraint (68), then there exists a state which is orthogonal to all of them. To construct such a state we simply take the first  $d - 1$  coherent states  $\{|z_1\rangle, \dots, |z_{d-1}\rangle\}$  (which form an undercomplete set because the space is  $d$  dimensional) and find a state  $|g\rangle$  which is orthogonal to them. The corresponding analytic function  $g(z)$  will have  $d$  zeros which will be the  $z_1, \dots, z_{d-1}$  and an extra one which has to obey the constraint (68) and therefore has to be  $z_d$ . Therefore  $|g\rangle$  will be orthogonal to  $|z_d\rangle$  also, and consequently the set of  $\{|z_1\rangle, \dots, |z_d\rangle\}$  is undercomplete.

## 7. Construction of the analytic representation of a state from its zeros

We have proved in the last section that for an arbitrary state  $|f\rangle$ , the analytic representation  $f(z)$  has  $d$  zeros in the cell  $S$  of equation (41). In this section we assume that the zeros  $z_1, z_2, \dots, z_d$  in the cell  $S$  are given (subject to the constraint of equation (68)) and we will construct the function  $f(z)$ . We note that some of the zeros might be equal to each other.

We first consider the product

$$Q(z) = \prod_{j=1}^d \Theta_3 \left[ (z - z_j + w) \left( \frac{\pi}{2d} \right)^{1/2} \lambda^{-1}; \frac{i}{\lambda^2} \right]; \quad w = \left( \frac{\pi d}{2} \right)^{1/2} (\lambda + \lambda^{-1}i). \quad (70)$$

It is easily seen that  $Q(z)$  has the given zeros. The ratio  $f(z)/Q(z)$  is entire function with no zeros and therefore it is the exponential of an entire function:

$$f(z) = Q(z) \exp(P(z)). \quad (71)$$

Taking into account the periodicity constraints of equation (51) we conclude that

$$\begin{aligned} P[z + (2\pi d)^{1/2}\lambda] &= P(z) + i2\pi K \\ P[z + i(2\pi d)^{1/2}\lambda^{-1}] &= P(z) + \frac{2\pi N}{\lambda^2} + i2\pi \Lambda. \end{aligned} \quad (72)$$

Here  $N$  is the integer entering the constraint of equation (68); and  $K, \Lambda$  are arbitrary integers. We have explained earlier that the growth of  $f(z)$  is of order 2. The order of  $Q(z)$  is 2; therefore, the  $P(z)$  is a polynomial of maximum possible degree 2. Equation (72) shows that in fact  $P(z)$  is

$$P(z) = - \left( \frac{2\pi}{d} \right)^{1/2} N \lambda^{-1} z i + C, \quad (73)$$

where  $C$  is a constant. Therefore,

$$f(z) = C' \exp \left[ - \left( \frac{2\pi}{d} \right)^{1/2} N \lambda^{-1} z i \right] Q(z), \quad (74)$$

where the constant  $C'$  is determined by the normalization condition.

## 8. Discussion

The harmonic oscillator formalism with phase space  $\mathbb{R} \times \mathbb{R}$  has been studied extensively in the literature. Equally interesting is quantum mechanics on a circle, with phase space  $\mathbb{S} \times \mathbb{Z}$  [22, 23]; and finite quantum systems, with phase space  $\mathbb{Z}_d \times \mathbb{Z}_d$ . Most of the results for physical systems on a circle or circular lattice (which is the case here) are intimately related to theta functions; and well-known mathematical results for theta functions can be used to derive interesting physical results for these systems.

In this paper we have introduced the transform of equation (17) between functions in  $\mathbb{R}$  and functions in  $\mathbb{Z}_d$ . The aim is to create a harmonic oscillator-like formalism in the context of finite systems. We have defined the analogue of number states for finite quantum systems in equation (25); and of coherent states in equation (30). The properties of these states have been discussed.

Using the coherent states we have defined the analytic representation of equation (50) in terms of theta functions. In this language we have studied displacements and the Heisenberg–Weyl group; and also more general transformations. Symplectic transformations are also important for these systems. Especially in the case where  $d$  is the power of a prime number, there are strong results (e.g., [9]). Further work is needed in order to express these results in the language of analytic representations used in this paper.

The analytic functions (50) have growth of order 2 and they have exactly  $d$  zeros in each cell  $S$ . If the zeros are given we can construct the analytic representation of the state using equation (74). Therefore we can describe the time evolution of a system through the paths of the  $d$  zeros of its analytic representation, in the cell  $S$ .

Based on the theory of zeros of analytic functions we have shown that a set of  $d + 1$  coherent states in the cell  $S$  is overcomplete; and a set of  $d - 1$  coherent states is undercomplete. A set of  $d$  coherent states in the cell  $S$  is complete if the constraint of equation (68) is violated; and undercomplete if the constraint of equation (68) is obeyed. These results are analogous to the ‘theory of von Neumann lattice’ in our context of finite quantum systems.

Our results use the powerful techniques associated with analytic representations in the context of finite systems.

## Appendix A

The normalization factor appearing in equation (17) is given by

$$\mathcal{N} = \sum_{m=0}^{d-1} \left\{ \sum_{w=-\infty}^{\infty} \psi^* \left[ x = \left( \frac{2\pi}{d} \right)^{1/2} \lambda(m + dw) \right] \right\} \left\{ \sum_{w'=-\infty}^{\infty} \psi \left[ x = \left( \frac{2\pi}{d} \right)^{1/2} \lambda(m + dw') \right] \right\}. \quad (\text{A.1})$$

The normalization factor appearing in equation (19) is given by

$$\mathcal{N}' = \sum_{m=0}^{d-1} \left\{ \sum_{w=-\infty}^{\infty} \tilde{\psi}^* \left[ p = \left( \frac{2\pi}{d} \right)^{1/2} \lambda^{-1}(m + dw) \right] \right\} \times \left\{ \sum_{w'=-\infty}^{\infty} \tilde{\psi} \left[ p = \left( \frac{2\pi}{d} \right)^{1/2} \lambda^{-1}(m + dw') \right] \right\}. \quad (\text{A.2})$$

We insert equation (18) into equation (A.2), and use equations (21) and (22) to prove that  $\mathcal{N}' = \lambda^2 \mathcal{N}$ .

## Appendix B

In this appendix we use the full Zak transform to introduce a family of  $d$ -dimensional Hilbert space  $\mathcal{H}(\sigma_1, \sigma_2)$  (with  $\mathcal{H} \equiv \mathcal{H}(0, 0)$ ). We generalize equation (17) into

$$\psi_m(\sigma_1, \sigma_2) = [\mathcal{N}(\sigma_1, \sigma_2)]^{-1/2} \sum_{w=-\infty}^{\infty} \exp(-2\pi i \sigma_1 w) \psi \left[ \left( \frac{2\pi}{d} \right)^{1/2} \lambda(m + \sigma_2 + dw) \right], \quad (\text{B.1})$$

where  $\mathcal{N}(\sigma_1, \sigma_2)$  is a normalization factor. The Hilbert space  $\mathcal{H}(\sigma_1, \sigma_2)$  is spanned by the states corresponding to  $\psi_m(\sigma_1, \sigma_2)$ . These spaces and the corresponding twisted boundary conditions of the wavefunctions have been studied in [24]. The Hilbert space  $H$  is isomorphic to the direct integral of all the  $\mathcal{H}(\sigma_1, \sigma_2)$  (with  $0 \leq \sigma_1 < 1, 0 \leq \sigma_2 < 1$ ). In this case, equation (17) can be inverted as follows:

$$\psi \left[ x = \left( \frac{2\pi}{d} \right)^{1/2} \lambda(m + \sigma_2 + dw) \right] = \int_0^1 [\mathcal{N}(\sigma_1, \sigma_2)]^{1/2} \psi_m(\sigma_1, \sigma_2) \exp(2\pi i \sigma_1 w) d\sigma_1. \quad (\text{B.2})$$

The formalism of this paper is valid for the space  $\mathcal{H} \equiv \mathcal{H}(0, 0)$ .

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